

# Poisson kernel and Green function of the ball in real hyperbolic spaces

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## Abstract

Let  $(X_t)_{t \geq 0}$  be the  $n$ -dimensional hyperbolic Brownian motion, that is the diffusion on the real hyperbolic space  $\mathbb{D}^n$  having the Laplace-Beltrami operator as its generator. The aim of the paper is to derive the formulas for the Gegenbauer transform of the Poisson kernel and the Green function of the ball for the process  $(X_t)_{t \geq 0}$ . Under some additional hypotheses we give the formulas for the Poisson kernel itself. In particular, we provide formulas in  $\mathbb{D}^4$  and  $\mathbb{D}^6$  spaces for the Poisson kernel and the Green function as well.

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# 1 Introduction

Investigation of the hyperbolic Brownian motion is an important and intensely developed topic in recent years (cf. [Y3], [BJ]). On the other hand, it is well known that the Poisson kernel for a region is a fundamental tool in harmonic analysis or probabilistic potential theory. In the classical situation of the Laplacian in  $\mathbb{R}^n$ , the exact formula for the kernel leads to many important results concerning behaviour of harmonic functions. Moreover, probabilistic potential theory uses Poisson kernel techniques to find solutions to the Schrödinger equation ([ChZ]). Availability of the exact formula for the kernel is often of crucial importance for the argument.

In the case of half-spaces in the model  $\mathbb{H}^n$  of real hyperbolic spaces (or, equivalently, the region bounded by a horocycle, in  $\mathbb{D}^n$ ) the form of the Fourier transform of the corresponding Poisson kernel is known for some time (see [Du] and [BCF]). However, in many applications the resulting Fourier-Hankel inversion formula is of little use. A satisfactory integral representation of the Poisson kernel in this case, along with the resulting analysis of the asymptotic behaviour, was given in [BGS].

The aim of this paper is to provide the Gegenbauer transform for the Poisson kernel and Green function of a ball in the real hyperbolic space  $\mathbb{D}^n$ . Next, we determine the formula for the Poisson kernel itself; first - as a series representation, and, in some cases - as an explicit integral formula. Results presented here are not that complete as in [BGS]; they depend on the properties of a hypergeometric function  $F_k$ , which appears quite naturally in the Gegenbauer transform of the Poisson kernel.

Although we are motivated here by the paper [W], where the Gegenbauer transform of the joint distribution of hitting time and hitting distributions for the ball in the case of classical Brownian motion in  $\mathbb{R}^n$  was found, there were very substantial difficulties to adapt the above approach to the present situation.

We also provide explicit formulas for Poisson kernel and Green function in  $\mathbb{D}^4$  and  $\mathbb{D}^6$ .

The paper is organized as follows. In Section 2, after some preliminaries, we apply stochastic calculus to write a "polar" decomposition of the hyperbolic Brownian motion on  $\mathbb{D}^n$ . It is the starting point to obtain, in Section 3, the basic formula for Gegenbauer coefficients of the cosine between the axis determined by the process and the starting point  $x \neq 0$  (see Theorem 3.1). We again use here the stochastic calculus; more specifically Feynmann-Kac technique and the relation with the appropriate Schrödinger equation. The series representation theorem of the Poisson kernel is the main result of this section.

In Section 4, in Theorem 4.1, we provide Gegenbauer coefficients for the Green function of the ball, restricted to a suitable sphere. In the next theorem we write series representation theorem for the Green function of the ball. Remark that although the Poisson kernel determines uniquely the Green function (by sweeping out formula), the substantial computational complexity forces us to find an alternative approach.

In Section 5 we try to describe an explicit integral representation of Poisson kernel. It turns out that all depends on the properties of the hypergeometric function  $F_k$ . Conditions sufficient to obtain the desired representation are collected in Conjectures 5.1 and 5.2. Checking the conditions imposed on the specific hypergeometric functions  $F_k$  we exhibit the exact form of this representation for  $\mathbb{D}^4$  and  $\mathbb{D}^6$ . In general, the validity of this conjecture depends on the location of roots of  $F_k$ , with respect to the (complex) variable  $k$ , which seems to be a difficult problem. We also provide explicit formulas for the Green function of the ball for  $\mathbb{D}^4$  and  $\mathbb{D}^6$ .

## 2 Preliminaries

We begin with some basic informations about hypergeometric functions and Gegenbauer polynomials, needed in the sequel. This part of the material is standard and can be found, e.g. in [E]. After that we identify Brownian motion in real hyperbolic spaces, in terms of Stochastic Differential Equations (SDE). We then discuss briefly properties of the heat kernel on hyperbolic spaces, following approach presented in the monograph of E. B. Davies [D]. In the end we obtain a kind of "polar" decomposition of the hyperbolic Brownian motion, in terms of SDE.

We denote here by  $(x, y)$  the standard inner product of  $x, y \in \mathbb{R}^n$  and by  $|x|$  the Euclidean length of a vector  $x$ . The sphere with center at 0 and the radius  $r$  is written as  $S_r = \{x \in \mathbb{R}^n : |x| = r\}$ . The  $(n-1)$ -dimensional spherical measure on  $S_r$  will be denoted by  $\sigma_r$ . Put

$$\omega_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}, \quad n = 1, 2, \dots$$

It is the total mass of the associated  $(n-1)$ -dimensional spherical measure of the unit sphere  $S_1$ . Note that for  $n = 1$ ,  $S_1$  is a two-point set and its 0-dimensional measure is equal to the counting measure by an accepted convention. For the rest of the paper we assume that  $n > 2$ .

We will denote by  $F(\alpha, \beta; \gamma; z)$  the hypergeometric function of variable  $z$  with parameters  $\alpha, \beta, \gamma$ . For  $|z| < 1$  and  $\alpha, \beta, \gamma \in \mathbb{C}$ ,  $\gamma \neq 0, -1, -2, \dots$  the function  $F$  is defined by the hypergeometric series

$$F(\alpha, \beta; \gamma; z) = \sum_{i=0}^{\infty} \frac{(\alpha)_i (\beta)_i}{(\gamma)_i i!} z^i,$$

where  $(\alpha)_i = \Gamma(\alpha + i)/\Gamma(\alpha)$  is the Pochhammer symbol. We shall supplement the definition in the case  $\alpha = -l$  and  $\gamma = -m$  where  $l = 0, 1, 2, \dots$ , and  $m = l, l+1, l+2, \dots$ . Then it is customary to define

$$F(-l, \beta; -m; z) = \sum_{i=0}^l \frac{(-l)_i (\beta)_i}{(-m)_i i!} z^i.$$

To simplify our notation we put  $\rho = \frac{n-2}{2}$  and define

$$F_k(z) = F(k, -\rho; k + \frac{n}{2}; z),$$

and for  $k > 0$  or  $n/2 \notin \mathbb{N}$

$$G_k(z) = F(-\rho, 2 - k - n; 2 - k - \frac{n}{2}; z).$$

When  $k = 0$  and  $n$  is an even number greater than 2 we put

$$G_0(z) = \rho \sum_{i=0, i \neq \rho}^{n-2} \binom{n-2}{i} \frac{(-1)^{i+1}}{i - \rho} z^i + \rho \binom{n-2}{\rho} (-1)^{\frac{n}{2}} z^{\rho} \log z.$$

The general solution of the hypergeometric equation

$$z(1 - z)u'' + (\gamma - (\alpha + \beta + 1)z)u' - \alpha\beta u = 0$$

for  $\alpha = k, \beta = -\rho$  and  $\gamma = k + \frac{n}{2}$  is given by

$$c_1 \cdot F_k(z) + \frac{c_2}{z^{k+\rho}} \cdot G_k(z), \quad (1)$$

where  $c_1, c_2$  are constants and  $k = 0, 1, 2, \dots$ . Observe that  $F_k(z)$  is bounded and  $z^{-k-\rho}G_k(z)$  (the second solution of (1)) is unbounded on the interval  $(0, a]$  for every  $a \in (0, 1)$ . It follows from the fact that the functions  $F_k(z), G_k(z)$  are continuous on  $[0, a]$ .

We also have

**Proposition 2.1.**

$$(k + \rho)F_k(z)G_k(z) + zF'_k(z)G_k(z) - zF_k(z)G'_k(z) = (k + \rho)(1 - z)^{n-2}. \quad (2)$$

*Proof.* Let denote by  $u(z)$  the function on the left-hand side of (2). Using the hypergeometric equations which are satisfied by  $F_k$  and  $G_k$  we find that  $(1 - z)u'(z) = (2 - n)u(z)$ . We can also show that  $u(0) = k + \rho$ . So the desired equality follows. The details are left to the reader.  $\square$

Gegenbauer's polynomial  $C_k^{(v)}(z)$  for integer value of  $k$  and  $v > 0$  is defined to be the coefficient of  $h^k$  in the Maclaurin expansion of  $(1 - 2zh + h^2)^{-v}$ , considered as a function of  $h$ . So we have

$$(1 - 2zh + h^2)^{-v} = \sum_{k=0}^{\infty} C_k^{(v)}(z)h^k, \quad |z| \leq 1, |h| < 1.$$

Observe that  $C_0^{(v)}(z) = 1$ , for all  $v > 0$ . For  $v = 0$  it is customary to take  $C_0^{(0)}(z) \equiv 1$ ,  $C_k^{(0)}(z) = \lim_{v \rightarrow 0} \frac{C_k^{(v)}(z)}{v} = \frac{2T_k(z)}{k}$ , where  $T_k$  is  $k$ th Chebyshev polynomial defined by  $T_k(\cos \phi) = \cos(n\phi)$ . One of generating functions for  $T_k$  is given by

$$\log(1 - 2zh + h^2)^{-1} = 2 \sum_{k=1}^{\infty} k^{-1} T_k(z)h^k.$$

We also have the following trigonometric expansion of  $C_k^{(v)}(\cos \phi)$ :

$$\Gamma(v)^2 C_k^{(v)}(\cos \phi) = \sum_{l=0}^k \frac{\Gamma(l+v)\Gamma(k-l+v)}{l!(k-l)!} \cdot e^{-i(k-2l)\phi},$$

which gives

$$|C_k^{(v)}(\cos \phi)| \leq C_k^{(v)}(1), \quad \phi \in [0, \pi]. \quad (3)$$

Note that  $C_k^{(v)}(1) = \Gamma(k+2v)/(k! \Gamma(2v))$ . We recall the orthogonal relations of Gegenbauer polynomials

$$\int_{-1}^1 C_k^{(v)}(x) C_l^{(v)}(x) (1-x^2)^{v-\frac{1}{2}} dx = \delta_{kl} \frac{2^{1-2v} \pi \Gamma(k+2v)}{k! (v+k) \Gamma(v)^2}.$$

In the special case for  $v = \rho$ , using the well known relation for gamma function

$$2^{2z-1} \Gamma(z) \Gamma(z + \frac{1}{2}) = \sqrt{\pi} \Gamma(2z),$$

we obtain

$$\int_{-1}^1 C_k^{(\rho)}(x) C_l^{(\rho)}(x) (1-x^2)^{\frac{n-3}{2}} dx = \delta_{kl} \frac{\rho}{k+\rho} \cdot C_k^{(\rho)}(1) \cdot \frac{\omega_{n-1}}{\omega_{n-2}}. \quad (4)$$

Note also that the polynomial  $C_k^{(v)}(z)$  is a solution of the Gegenbauer differential equation

$$(z^2 - 1)\omega'' + (2v + 1)z\omega' - k(k + 2v)\omega = 0. \quad (5)$$

Now, we introduce some basic information about measures (functions) on spheres and their Gegenbauer transforms. For more details see [W] and [E].

We say that a finite Borel measure  $\mu(\cdot)$  on sphere  $S_r$  is *axially symmetric* (AS) with axis  $x \in \mathbb{R}^n$  if  $\mu(UA) = \mu(A)$  for each Borel set  $A$  and each orthogonal transformation  $U$  such that  $Ux = x$ . The definition of AS functions is similar. For AS measures we define its Gegenbauer coefficient  $\widehat{\mu}_k$  by

$$C_k^{(\rho)}(1)\widehat{\mu}_k = \int_{S_r} C_k^{(\rho)}(\cos \theta) \mu(dy),$$

where  $\theta = \angle x0y$  if  $x \neq 0$  and  $\theta = \angle u0y$ , for arbitrary but fixed nonzero vector  $u$ , in case  $x = 0$ . We will use the following property of Gegenbauer transform:

**Theorem 2.2.** *The AS measure  $\mu$  is uniquely determined by its transform  $\{\widehat{\mu}_k\}_{k=0}^\infty$ .*

Clearly, the same is true for AS functions, i.e. any AS function is uniquely determined by its Gegenbauer transform.

Consider the ball model of the  $n$ -dimensional real hyperbolic space

$$\mathbb{D}^n = \{x \in \mathbb{R}^n : |x| < 1\}, \quad n > 2.$$

The Riemannian metric and the distance formula are given by

$$ds^2 = \frac{|dx|^2}{(1 - |x|^2)^2}, \quad (6)$$

$$\cosh(2d(x, y)) = 1 + \frac{2|x - y|^2}{(1 - |x|^2)(1 - |y|^2)}.$$

The canonical (hyperbolic) volume element is given by

$$dV_n = \frac{dx}{(1 - |x|^2)^n}.$$

Consider the following system of SDE:

$$\frac{dX_k(t)}{1 - |X(t)|^2} = dB_k(t) + 2(n - 2)X_k(t)dt, \quad k = 1, \dots, n.$$

For  $f \in \mathcal{C}^2$  by Itô Formula we obtain

$$\begin{aligned} f(X_t) - f(X_0) &= \sum_{i=1}^n \int_0^t \frac{\partial f}{\partial x_i} dX_i(s) + \frac{1}{2} \sum_{i=1}^n \int_0^t \frac{\partial^2 f}{\partial x_i^2} d\langle X_i \rangle(s) \\ &= \sum_{i=1}^n \int_0^t \frac{\partial f}{\partial x_i} (1 - |X(s)|^2) dB_i(s) + 2(n - 2) \sum_{i=1}^n \int_0^t \frac{\partial f}{\partial x_i} (1 - |X(s)|^2) X_i(s) ds \\ &\quad + \sum_{i=1}^n \int_0^t \frac{\partial^2 f}{\partial x_i^2} (1 - |X(s)|^2)^2 ds. \end{aligned}$$

Here  $B = (B_1, \dots, B_n)$  denotes the standard Brownian motion with scaling such that  $EB_k^2(t) = 2t$ ,  $k = 1, \dots, n$ . Thus, the generator of the process  $X(t)$  determined by the above system of SDE is given by

$$\Delta_B = (1 - |x|^2)^2 \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + 2(n - 2)(1 - |x|^2) \sum_{i=1}^n x_i \frac{\partial}{\partial x_i},$$

and is the canonical Laplace - Beltrami operator associated with the Riemannian metric (6). Since the Laplace - Beltrami operator commutes with isometries acting on  $\mathbb{D}^n$ , the heat kernel  $k_n$ , i.e. the transition density of the hyperbolic Brownian motion is a function of the (hyperbolic) distance  $d(x, y)$ . Fix  $a \in \mathbb{D}^n$  and denote  $\rho(x) = d(a, x)$ . We have the following explicit form of the heat kernel  $k_2(t, \rho)$  on the hyperbolic disc  $\mathbb{D}^2$  (see [D]):

$$k_2(t, \rho) = 2^{\frac{3}{2}}(4\pi t)^{-3/2}e^{-t} \int_{\rho}^{\infty} \frac{se^{-s^2/4t} ds}{(\cosh(2s) - \cosh(2\rho))^{1/2}},$$

while on  $\mathbb{D}^3$  we obtain

$$k_3(t, \rho) = 2(4\pi t)^{-3/2}e^{-4t-\rho^2/4t} \frac{\rho}{\sinh(2\rho)}.$$

In higher dimensions we have the following recursion formula:

$$k_n(t, \rho) = \sqrt{2}e^{(2n-1)t} \int_{\rho}^{\infty} \frac{k_{n+1}(t, \lambda) \sinh(2\lambda) d\lambda}{(\cosh(2\lambda) - \cosh(2\rho))^{1/2}}. \quad (7)$$

We now show that the hyperbolic Brownian motion on  $\mathbb{D}^2$  is *transitive* for  $n \geq 2$  (see, e.g. [Ch]), a fact which is widely known; we include it for the reader's convenience. For this purpose it is enough to show that the potential  $U_n(z) < \infty$ , for almost all  $0 < z \in \mathbb{R}$ . A direct computation on  $\mathbb{D}^2$  yields

$$U_2(\rho) = \int_0^{\infty} k_2(t, \rho) dt = \sqrt{2}(4\pi)^{-1} \ln \coth(\rho).$$

Note that  $k_n$  is the density function with respect to the canonical hyperbolic volume element  $dV_n$ .

The hyperbolic Brownian motion process on  $\mathbb{D}^n$  with the measure  $dV_n$  as the reference measure fits into context of the so-called "dual processes" and their potential theory (see [BG], ch. VI). From this theory it follows, in particular, that single points in  $\mathbb{D}^n$  are polar, whenever we show that the potential kernel  $U_n(\rho) < \infty$ , for  $\rho > 0$ . Indeed, by Proposition 3.5, Ch. II, in [BG] we obtain that  $\{a\} = \{U_n(\rho) = \infty\}$  is polar if and only if it is null (of potential 0). This last condition is obviously satisfied whenever there exists almost everywhere finite potential kernel  $U_n$ . Thus, the Brownian motion on  $\mathbb{D}^2$  does not hit single points, a.s. For higher dimensions we use recursion formula (7) to obtain:

$$\begin{aligned} \int_{\rho}^{\infty} \frac{\sinh(2\lambda) U_{n+1}(\lambda)}{(\cosh(2\lambda) - \cosh(2\rho))^{1/2}} d\lambda &= \int_{\rho}^{\infty} \frac{\sinh(2\lambda)}{(\cosh(2\lambda) - \cosh(2\rho))^{1/2}} \int_0^{\infty} k_{n+1}(t, \lambda) dt d\lambda \\ &= 2^{-1/2} \int_0^{\infty} e^{-(2n-1)t} k_n(t, \lambda) dt \leq 2^{-1/2} \int_0^{\infty} k_n(t, \lambda) dt = 2^{-1/2} U_n(\lambda). \end{aligned}$$

By induction, we obtain for  $n \geq 2$

$$U_n(\rho) < \infty \quad \text{a.e.} \quad (8)$$

For the hyperbolic Brownian motion  $X_t$  let  $x = X(0)$  be the starting point. We define the following two processes :

$$R_t = \sum_{i=1}^n X_i^2(t) = |X_t|^2, \quad \Phi_t = \cos \angle x0X_t = \frac{(x, X(t))}{|x||X(t)|}. \quad (9)$$

We always assume that  $x \neq 0$ . Then also  $X(t) \neq 0$ , a.s. (see (8)). Thus,  $\Phi_t$  is well defined almost surely.

**Proposition 2.3.** *The process  $(R(t), \Phi(t))$  satisfies the following system of stochastic differential equations:*

$$\begin{cases} dR(t) = 2(1 - R(t)) \left( \sqrt{R(t)} dW_1(t) + ((n-4)R(t) + n) dt \right) \\ d\Phi(t) = (1 - R(t)) \left( \frac{1 - \Phi^2(t)}{R(t)} \right)^{\frac{1}{2}} dW_2(t) - (n-1) \frac{(1-R(t))^2}{R(t)} \Phi(t) dt, \end{cases} \quad (10)$$

where  $W_1(t), W_2(t)$  are independent Brownian motions on  $\mathbb{R}$  with variation  $2t$ .

*Proof.* We define

$$\begin{aligned} W_1(t) &= \int_0^t \frac{\sum_{i=1}^n X_i(s) dB_i(s)}{\sqrt{R(s)}}, \\ W_2(t) &= \int_0^t \mathbf{1}_{(\Phi(s)<1)} \left( \frac{R(s)}{1 - \Phi^2(s)} \right)^{\frac{1}{2}} \sum_{i=1}^n \left( \frac{x_i}{|x||X(s)|} - \frac{X_i(s)(x, X(s))}{|x||X(s)|^3} \right) dB_i(s) \\ &+ \int_0^t \mathbf{1}_{(\Phi(s)=1)} d\tilde{B}(s), \end{aligned}$$

where  $\tilde{B}(t)$  is classical Brownian motion on  $\mathbb{R}$  (with variation  $2t$ ) such that  $B_1(t), \dots, B_2(t), \tilde{B}(t)$  are independent. It is clear from the definitions and property of the Itô integral that  $W_1(t), W_2(t)$  are local martingales. We also have

$$\begin{aligned} d\langle W_1, W_1 \rangle(t) &= \frac{\sum_{i=1}^n X_i^2(t)}{R(t)} 2dt = 2dt, \\ d\langle W_2, W_2 \rangle(t) &= \mathbf{1}_{(\Phi(s)<1)} \frac{R(t)}{1 - \Phi^2(t)} \sum_{i=1}^n \left( \frac{x_i^2}{|x|^2|X(t)|^2} - \frac{2x_i X_i(t)(x, X(t))}{|x|^2|X(t)|^4} \right. \\ &\quad \left. + \frac{X_i^2(t)(x, X(t))^2}{|x|^2|X(t)|^6} \right) 2dt + \mathbf{1}_{(\Phi(t)=1)} 2dt = 2dt, \\ d\langle W_1, W_2 \rangle(t) &= \mathbf{1}_{(\Phi(t)<1)} (1 - \Phi^2(t))^{-\frac{1}{2}} \sum_{i=1}^n \left( \frac{x_i X_i(t)}{|x||X(t)|} - \frac{X_i^2(t)(x, X(t))}{|x||X(t)|^3} \right) 2dt = 0. \end{aligned}$$

So  $W_1(t), W_2(t)$  are independent Brownian motions on  $\mathbb{R}$ . Using (2) and Itô Formula we get

$$\begin{aligned} dR(t) &= \sum_{i=1}^n 2X_i(t) dX_i(t) + \sum_{i=1}^n d\langle X_i \rangle(t) \\ &= 2(1 - |X(t)|^2) \left( \sum_{i=1}^n X_i(t) dB_i(t) + 2(n-2) \sum_{i=1}^n X_i^2(t) dt + n(1 - |X(t)|^2) dt \right) \\ &= 2(1 - R(t)) \left( \sqrt{R(t)} dW_1(t) + ((n-4)R(t) + n) dt \right). \end{aligned}$$

For the function  $g(y) = \frac{(x,y)}{|x||y|}$  we have

$$\frac{\partial g}{\partial y_i} = \frac{x_i}{|x||y|} - \frac{(x,y)y_i}{|x||y|^3}, \quad \frac{\partial^2 g}{\partial y_i^2} = -2\frac{x_i y_i}{|x||y|^3} - \frac{(x,y)}{|x||y|^3} + 3\frac{(x,y)y_i^2}{|x||y|^5}$$

and

$$\sum_{i=1}^n \frac{\partial^2 g}{\partial y_i^2} = -(n-1) \frac{(x, y)}{|x||y|^3}.$$

So Itô Formula gives

$$d\Phi(t) = (1 - R(t)) \left( \frac{1 - \Phi^2(t)}{R(t)} \right)^{\frac{1}{2}} dW_2(t) - (n-1) \frac{(1 - R(t))^2}{R(t)} \Phi(t) dt.$$

□

The following lemma will be useful in the next sections. The proof is standard and is omitted.

**Lemma 2.4.** *Let  $W = (W_1, W_2)$  be the 2-dimensional Brownian motion. Suppose that  $\Psi(t)$  is a real measurable process adapted to  $\mathcal{F}(W(t))$  such that for every  $t > 0$ ,  $E^x[\int_0^t \Psi^2(s) ds] < \infty$ ,  $x \in \mathbb{R}^2$ , and  $Y(t)$  be square integrable process adapted to  $\mathcal{F}(W_1(t))$ . Then we have*

$$E^x[Y(t) \int_0^t \Psi(s) dW_2(s)] = 0.$$

### 3 Poisson kernel of the ball

Consider a ball  $D = \{x \in \mathbb{D}^n : |x| < r\}$  for some fixed  $0 < r < 1$ . Recall, using the distance formula, that  $D$  is a hyperbolic ball with center at 0 and the radius  $\frac{1}{2} \log \frac{1+r}{1-r}$ . Define

$$\tau_D = \inf\{t \geq 0 : X(t) \notin D\} = \inf\{t \geq 0 : |X(t)| \geq r\}.$$

By  $P_r(x, y)$ ,  $x \in D$ ,  $y \in \partial D = S_r$  we denote the Poisson kernel of  $D$ , i.e. the density of the harmonic measure defined by  $\mu_x(A) = P^x(X(\tau_D) \in A)$  for every Borel subset  $A \subset S_r$  and  $x \in D$ .

Let  $U$  be an orthogonal transformation that leaves the starting point  $x$  fixed. It is easy to see that the processes  $X_t$  and  $U^{-1}(X_t)$  have the same distribution under  $P^x$ . Thus, for each Borel set  $A \subset S_r$  we get

$$P^x(X(\tau_D) \in U(A)) = P^x(U^{-1}(X(\tau_D)) \in A) = P^x(X(\tau_D) \in A).$$

Consequently, the harmonic measure is AS measure with axis  $x$ . It follows that the Poisson kernel  $P_r(x, \cdot)$ , as the density function of the harmonic measure, is AS function on  $S_r$  with axis  $x$ .

Below we identify the Poisson kernel in terms of its Gegenbauer transform. We give explicit formula for its Gegenbauer coefficients.

**Theorem 3.1.** *For  $|x| < r$  we have*

$$\frac{E^x C_k^{(\rho)}(\Phi(\tau_D))}{C_k^{(\rho)}(1)} = \left( \frac{|x|}{r} \right)^k \frac{F_k(|x|^2)}{F_k(r^2)}, \quad (11)$$

with  $k = 0, 1, 2, \dots$

*Proof.* We define

$$\begin{aligned} V(t) &= \exp\left(\int_0^t q(R(s))ds\right), \\ Z(t) &= \varphi(\Phi(t))V(t) = C_k^{(\rho)}(\Phi(t)) \exp\left(\int_0^t q(R(s))ds\right), \end{aligned} \quad (12)$$

where  $q(x) = k(k+n-2)\frac{(1-x)^2}{x}$  and  $\varphi(z) = C_k^{(\rho)}(z)$  is the Gegenbauer polynomial introduced in Preliminaries. Observe that the process  $V$  has bounded variation sample paths so we obtain

$$dV(t) = q(R(t))V(t)dt.$$

The Itô formula yields:

$$\begin{aligned} dZ(t) &= \varphi'(\Phi_t)V(t)d\Phi(t) + \varphi(\Phi_t)dV(t) + \frac{1}{2}\varphi''(\Phi_t)V(t)d\langle\Phi\rangle(t) \\ &= \varphi'(\Phi_t)V(t)(1-R(t))\left(\frac{1-\Phi_t^2}{R(t)}\right)^{\frac{1}{2}}dW_2(t) \\ &\quad + V(t)\frac{(1-R(t))^2}{R(t)}\left((1-\Phi_t^2)\varphi''(\Phi_t) - (n-1)\Phi_t\varphi'(\Phi_t) + k(k+n-2)\varphi(\Phi_t)\right)dt. \end{aligned}$$

Using (5) we obtain

$$dZ(t) = \varphi'(\Phi_t)V(t)(1-R(t))\left(\frac{1-\Phi_t^2}{R(t)}\right)^{\frac{1}{2}}dW_2(t).$$

Denote

$$T_n = \inf\{t > 0; R(t) \leq 1/n\}.$$

Since  $R(t) > 0$  we obtain  $T_n \rightarrow \infty$ .

From the definition of  $Z_t$  and the previous equality we obtain

$$\begin{aligned} E^x C_k^{(\rho)}(\Phi(t \wedge T_n \wedge \tau_D)) &= E^x \left( Z(t \wedge T_n \wedge \tau_D) \exp\left(-\int_0^{t \wedge T_n \wedge \tau_D} q(R(s))ds\right) \right) \\ &= C_k^{(\rho)}(1) E^{|x|^2} \exp\left(-\int_0^{t \wedge T_n \wedge \tau_D} q(R(s))ds\right) + E^x \left( V^{-1}(t \wedge T_n \wedge \tau_D) \int_0^{t \wedge T_n} \Psi(s) dW_2(s) \right), \end{aligned}$$

where

$$\Psi(t) = \mathbf{1}_{\{t \wedge \tau_D\}} \varphi'(\Phi_t)V(t)(1-R(t))\left(\frac{1-\Phi_t^2}{R(t)}\right)^{\frac{1}{2}}.$$

Since  $\tau_D$  and  $T_n$  depend only on  $R_t$  (i.e.  $W_1$ ) we can use Lemma 2.4 to show that the last expectation is equal to zero. Moreover, since  $t \wedge T_n \wedge \tau_D$  tends to  $\tau_D$  as  $t \rightarrow \infty$  and  $n \rightarrow \infty$ , and the functions  $C_k^{(\rho)}(\Phi(t \wedge T_n \wedge \tau_D))$  and  $\exp\left(-\int_0^{t \wedge T_n \wedge \tau_D} q(R(s))ds\right)$  are bounded by a constant, we obtain (using dominated convergence theorem) that

$$E^x C_k^{(\rho)}(\Phi(\tau_D)) = C_k^{(\rho)}(1) E^{|x|^2} e_{-q}(\tau_D), \quad (13)$$

where  $e_{-q}(\tau_D) = \exp\left(-\int_0^{\tau_D} q(R(s))ds\right)$ .

Observe that the function  $\phi(y) = E^y e_{-q}(\tau_D)$  is by definition the gauge function for the Schrödinger operator based on the generator of the process  $R_t$  and the potential  $(-q)$ . According to the general theory (see [ChZ]) the function  $\phi$  is the solution of the appropriate Schrödinger equation. Using (10) we obtain the following formula for the generator of  $R_t$ :

$$4(1-x)^2 x \frac{d^2}{dx^2} + 2(1-x)((n-4)x + n) \frac{d}{dx}.$$

Hence,  $\phi$  satisfies the following equation

$$4(1-x)^2 x \phi''(x) + 2(1-x)((n-4)x + n) \phi'(x) - k(k+n-2) \frac{(1-x)^2}{x} \phi(x) = 0 \quad (14)$$

on  $(0, r^2]$ . Let  $\phi(x) = x^{\frac{k}{2}} g(x)$ . Then  $\phi'(x) = \frac{k}{2} x^{\frac{k-2}{2}} g(x) + x^{\frac{k}{2}} g'(x)$  and  $\phi''(x) = \frac{k(k-2)}{4} x^{\frac{k-4}{2}} g(x) + kx^{\frac{k-2}{2}} g'(x) + x^{\frac{k}{2}} g''(x)$  and consequently (14) reads as

$$x(1-x)g''(x) + (k + \frac{n}{2} - (k - \frac{n-2}{2} + 1)x)g'(x) + k\frac{n-2}{2}g(x) = 0. \quad (15)$$

This is the hypergeometric equation with parameters  $\alpha = k$ ,  $\beta = -\rho$  and  $\gamma = k + \frac{n}{2}$ . The general solution of (15) is given by (1) and we infer that

$$\phi(x) = x^{\frac{k}{2}} \left( c_1 \cdot F_k(x) + \frac{c_2}{x^{\rho+k}} G_k(x) \right).$$

By definition  $\phi(x)$  is bounded and  $\phi(r^2) = 1$ . Therefore  $c_2 = 0$  and the other condition gives the normalizing constant

$$c_1 = \frac{1}{r^k F_k(r^2)}.$$

This completes the proof.  $\square$

**Theorem 3.2 (Poisson kernel formula).** *For  $|x| < r$ ,  $|y| = r$  we have*

$$P_r(x, y) = \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}} r^{n-1}} \sum_{k=0}^{\infty} \frac{k+\rho}{\rho} \left( \frac{|x|}{r} \right)^k \frac{F_k(|x|^2)}{F_k(r^2)} C_k^{(\rho)}(\cos \theta). \quad (16)$$

*Proof.* It is easy to see that the terms of the series (16) are continuous functions of  $y$ . We will show that the series is uniformly convergent on the sphere  $S_r$ . Indeed, we have for every  $|x| < 1$

$$|F_k(x)| \leq \sum_{j=0}^{\infty} \frac{(k)_j}{(k + \frac{n}{2})_j} \frac{|(-\rho)_j|}{j!} |x|^j \leq \sum_{j=0}^{\infty} \frac{|(-\rho)_j|}{j!} |x|^j < \infty$$

and using the above inequality and uniform convergence we obtain that

$$\lim_{k \rightarrow \infty} F_k(x) = \sum_{j=0}^{\infty} \frac{(-\rho)_j}{j!} x^j = (1-x)^\rho.$$

Moreover, as a consequence of the relation  $E^x(e_{-q}(\tau)) = \left(\frac{|x|}{r}\right)^k \frac{F_k(|x|^2)}{F_k(r^2)}$ , we get  $F_k(r^2) \neq 0$  for every  $k \in \mathbb{N}$ . Thus there exists a constant  $c = c(r, |x|)$  such that

$$\left| \frac{F_k(|x|^2)}{F_k(r^2)} \right| \leq c.$$

From this and (3) we obtain

$$\left| \sum_{k=0}^{\infty} \frac{k+\rho}{\rho} \left( \frac{|x|}{r} \right)^k \frac{F_k(|x|^2)}{F_k(r^2)} C_k^{(\rho)}(\cos \theta) \right| < c \sum_{k=0}^{\infty} \frac{k+\rho}{\rho} \left( \frac{|x|}{r} \right)^k C_k^{(\rho)}(1) < \infty. \quad (17)$$

Applying the formula for  $C_k^{(\rho)}(1)$  we obtain  $C_{k+1}^{(\rho)}(1)/C_k^{(\rho)}(1) = (n+k-1)/(k+1)$ . Using Ratio Criterion for convergence of power series we infer that the above series is convergent for  $|x| < r$ . Consequently, the series (16) represents continuous function of  $y$  which is axially symmetric. Using (4) and (17) we can check that the Gegenbauer coefficient of that function is the same as the Gegenbauer coefficient of the Poisson kernel computed in Theorem 3.1. The desired equality follows from Theorem 2.2.  $\square$

## 4 Green function of the ball

Let  $(X_t^D, P_t^D)$  be the hyperbolic Brownian motion killed at the boundary  $\partial D$ . The Green function of  $D$  is defined by

$$G_D(x, y) = \int_0^\infty p_t^D(x, y) dt \quad x, y \in D, \quad x \neq y.$$

where  $p_t^D(x, y)$  is the density function of  $P_t^D$ . Similar arguments as in the case of the Poisson kernel show that

$$P_t^D(U(A)) = P^x(t < \tau_D; X_t \in U(A)) = P^x(t < \tau_D; X_t \in A) = P_t^D(A),$$

for each orthogonal transformation  $U$  such that  $Ux = x$ . Thus, the Green function  $G_D(x, \cdot)$  as a function on the sphere  $S_R$ ,  $0 < R < r$ , is AS function with axis  $x$ . Recall that its Gegenbauer coefficient  $(\widehat{G_D})_k(x, R)$  is defined by

$$C_k^{(\rho)}(1)(\widehat{G_D})_k(x, R) = \frac{1}{\omega_{n-1} R^{n-1}} \int_{S_R} C_k^{(\rho)}(\cos \theta) G_D(x, y) d\sigma_R(y), \quad x \neq 0.$$

Here  $\cos \theta = \frac{(x, y)}{|x||y|}$ . Observe also that  $(\widehat{G_D})_k(x, R) = (\widehat{G_D})_k(|x|, R) = (\widehat{G_D})_k(R, |x|)$ . So from now on we write  $(\widehat{G_D})_k(|x|, R)$ . Moreover, in view of the above-mentioned symmetry, it is enough to determine  $(\widehat{G_D})_k(|x|, R)$  for  $|x| < R$ .

**Theorem 4.1.** *For  $|x| < R < r$  we have*

$$(\widehat{G_D})_k(|x|, R) = C_n \frac{\rho}{k+\rho} \cdot |x|^k F_k(|x|^2) R^k \left( \frac{G_k(R^2)}{R^{2k+2\rho}} - \frac{G_k(r^2)}{r^{2k+2\rho}} \cdot \frac{F_k(R^2)}{F_k(r^2)} \right),$$

where  $C_n = \frac{\Gamma(\frac{n}{2}-1)}{4\pi^{\frac{n}{2}}}$ .

*Proof.* Let  $(R_t^{\tilde{D}}, P_t^{\tilde{D}})$  be the process  $R_t$  killed at the boundary of  $\tilde{D} = \{x \in (0, 1) : x < r^2\}$ . Observe that  $R_t^{\tilde{D}} = |X_t^D|^2$  and  $\tau_{\tilde{D}} = \tau_D$ . Let  $Z_t$  be the process defined in (12). The same arguments as in the proof of Theorem 3.1 show that

$$\begin{aligned} E^x[t < T_n \wedge \tau_D; h(|X_t|^2)\varphi(\Phi_t)] &= E^x[t < T_n \wedge \tau_D; h(|X_t|^2) Z_t \exp(-\int_0^t q(R_s) ds)] \\ &= C_k^{(\rho)}(1) E^{|x|^2}[t < T_n \wedge \tau_D; h(R_t) e_{-q}(t)]. \end{aligned}$$

for every bounded Borel function  $h$ . We have, as before,  $T_n \rightarrow \infty$  so letting  $n \rightarrow \infty$  we obtain

$$E^x[h(|X_t^D|^2)\varphi(\Phi_t^D)] = C_k^{(\rho)}(1)E^{|x|^2}[h(R_t^{\tilde{D}})e_{-q}(t)].$$

It means that

$$(P_t^D \tilde{h} \tilde{\varphi})(x) = C_k^{(\rho)}(1)T_t^{\tilde{D}}h(|x|^2),$$

where  $\tilde{h}(x) = h(|x|^2)$ ,  $\tilde{\varphi}(y) = \tilde{\varphi}_x(y) = \varphi(\frac{(x,y)}{|x||y|})$ , and  $\{T_t^{\tilde{D}}\}$  is the killed at  $\partial\tilde{D} = \{r^2\}$  Feynman-Kac semigroup based on the process  $R_t$  with the potential  $(-q)$ . Consequently,

$$\int_0^\infty (P_t^D \tilde{h} \tilde{\varphi})(x) dt = C_k^{(\rho)}(1) \int_0^\infty T_t^{\tilde{D}}h(|x|^2) dt,$$

and

$$(G_D \tilde{h} \tilde{\varphi})(x) = C_k^{(\rho)}(1) V_{\tilde{D}}h(|x|^2), \quad (18)$$

where  $G_D$  is the Green operator for the process  $X_t$  and the set  $D$  and  $V_{\tilde{D}}$  is the Green operator for the semigroup  $\{T_t\}$  and the set  $\tilde{D}$ .

Integrating in polar coordinates we have

$$\begin{aligned} (G_D \tilde{h} \tilde{\varphi})(x) &= \int_D C_k^{(\rho)}(\cos \theta) G_D(x, y) h(|y|^2) dy \\ &= \int_0^r h(R^2) \left\{ \int_{S_R} C_k^{(\rho)}(\cos \theta) G_D(x, y) d\sigma_R(y) \right\} dR, \end{aligned} \quad (19)$$

and

$$V_{\tilde{D}}h(|x|^2) = \int_0^{r^2} h(y) V_{\tilde{D}}(|x|^2, y) dy = \int_0^r h(R^2) V_{\tilde{D}}(|x|^2, R^2) 2R dR. \quad (20)$$

Now, comparing (19), (20), (18) and applying the standard continuity arguments, we obtain

$$\widehat{(G_D)_k}(|x|, R) = \frac{2}{\sigma_{n-1} R^{n-2}} \cdot V(|x|^2, R^2). \quad (21)$$

Since the  $q$ -Green function  $V_{\tilde{D}}(y, R^2)$  is  $q$ -harmonic in  $y$ , for  $0 < y < R$  and for  $R < y < r$  (see [ChZ]), we obtain that  $\phi(y) = V_{\tilde{D}}(y, R^2)$  is a solution of the same Schrödinger equation as in the case of the gauge function in the proof of Theorem 3.1 and the same computations as before lead to the hypergeometric equation (15). Thus, on each of the intervals  $(0, R)$ ,  $(R, r]$ , we can write

$$V_{\tilde{D}}(|x|^2, R^2) = |x|^k \left( c_1(k, R^2) F_k(|x|^2) + \frac{c_2(k, R^2)}{|x|^{2k+2\rho}} G_k(|x|^2) \right).$$

We have  $\lim_{|x|^2 \rightarrow r^2} V_{\tilde{D}}(|x|^2, R^2) = 0$ . Therefore, for  $R < |x| < r$

$$V_{\tilde{D}}(|x|^2, R^2) = c(k, R^2) \cdot |x|^k \left( \frac{G_k(|x|^2)}{|x|^{2k+2\rho}} - \frac{G_k(r^2)}{r^{2k+2\rho}} \cdot \frac{F_k(|x|^2)}{F_k(r^2)} \right).$$

As a consequence of the symmetry property of  $\widehat{(G_D)_k}(|x|, R)$  it follows easily that for  $|x| < R$  we have

$$\begin{aligned} \widehat{(G_D)_k}(|x|, R) &= \widehat{(G_D)_k}(R, |x|) \\ &= \frac{2}{\omega_{n-1} |x|^{n-2}} \cdot V_{\tilde{D}}(R^2, |x|^2) \\ &= \frac{2 c(k, |x|^2)}{\omega_{n-1} |x|^{n-2}} \cdot R^k \left( \frac{G_k(R^2)}{R^{2k+2\rho}} - \frac{G_k(r^2)}{r^{2k+2\rho}} \cdot \frac{F_k(R^2)}{F_k(r^2)} \right). \end{aligned}$$

To find  $c(k, |x|^2)$  we use relation between Green function and Poisson kernel. Due to Green's theorem on hyperbolic space  $\mathbb{D}^n$  we can obtain the Poisson kernel by differentiating the Green function in the normal direction (see [C], page 174, theorem 8), i.e. for any  $y \in S_1$  we get  $-\frac{d}{dR} \Big|_{R=r} G_D(x, Ry) = (1-r^2)^{n-2} P_r(x, ry)$ . Note that the factor  $(1-r^2)^{n-2}$  appears because we consider the Poisson kernel as a density with respect to the Lebesgue measure  $\sigma_r$  (not with respect to the hyperbolic measure on the sphere). By bounded convergence theorem and (11) we obtain

$$\begin{aligned}
C_k^{(\rho)}(1) \omega_{n-1} \cdot \left( -\frac{d}{dR} \Big|_{R=r} \right) (\widehat{G_D})_k(|x|, R) &= \left( -\frac{d}{dR} \Big|_{R=r} \right) \int_{S_1} C_k^{(\rho)}(\cos \theta) G_D(x, Ry) d\sigma_1(y) \\
&= \int_{S_1} \left( -\frac{d}{dR} \Big|_{R=r} \right) C_k^{(\rho)}(\cos \theta) G_D(x, Ry) d\sigma_1(y) \\
&= (1-r^2)^{n-2} \int_{S_1} C_k^{(\rho)}(\cos \theta) P_r(x, ry) d\sigma_1(y) \\
&= \frac{(1-r^2)^{n-2}}{r^{n-1}} \int_{S_r} C_k^{(\rho)}(\cos \theta) P_r(x, y) d\sigma_r(y) \\
&= \frac{(1-r^2)^{n-2}}{r^{n-1}} C_k^{(\rho)}(1) \cdot \frac{|x|^k F_k(|x|^2)}{r^k F_k(r^2)}.
\end{aligned}$$

On the other hand, using (2), we have

$$-\frac{d}{dR} \Big|_{R=r} \left( \frac{G_k(R^2)}{R^{2k+2\rho}} - \frac{G_k(r^2)}{r^{2k+2\rho}} \cdot \frac{F_k(R^2)}{F_k(r^2)} \right) = \frac{2(1-r^2)^{n-2}}{r^{n-1}} \cdot \frac{k+\rho}{r^{2k} F_k(r^2)}.$$

Thus

$$\frac{2c(k, |x|^2)}{\omega_{n-1} |x|^{n-2}} = C_n \cdot \frac{\rho}{k+\rho} \cdot |x|^k F_k(|x|^2).$$

The proof is completed.  $\square$

**Theorem 4.2 (Green function formula).** *For  $|x| < |y|$  we have*

$$G_D(x, y) = C_n \sum_{k=0}^{\infty} |x|^k F_k(|x|^2) |y|^k \left( \frac{G_k(|y|^2)}{|y|^{2k+2\rho}} - \frac{G_k(r^2)}{r^{2k+2\rho}} \cdot \frac{F_k(|y|^2)}{F_k(r^2)} \right) C_k^{(\rho)}(\cos \theta). \quad (22)$$

*Proof.* We will consider both functions in (22) as a function of  $y$  on the sphere  $S_R$  where  $R = |y|$ . We now prove, under the hypothesis  $|x| < |y|$ , that the series is uniformly convergent on  $S_R$ . Indeed, it is easy to see that there exist constants  $c_1 = c_1(n, z)$  such that  $|G_k(z)| \leq c_1$ . Moreover, we already know that  $|F_k(z)| \leq c_2$  and  $|F_k(R^2)/F_k(r^2)| \leq c_3$  where  $c_2, c_3$  do not depend on  $k$ . Thus the norm of the series is bounded by

$$c \left[ |y|^{2-n} \sum_{k=0}^{\infty} \left( \frac{|x|}{|y|} \right)^k C_k^{(\rho)}(1) + r^{2-n} \sum_{k=0}^{\infty} \left( \frac{|x||y|}{r^2} \right)^k C_k^{(\rho)}(1) \right] < \infty.$$

Since the terms of the series are continuous functions of  $y$ , the series represents continuous function on the sphere  $S_R$  which is AS. Its Gegenbauer coefficients are the same as the coefficients of the Green function computed in Theorem 4.1. This completes the proof.  $\square$

## 5 Examples

In this section we provide an explicit integral formula for the Poisson kernel, under some additional conditions imposed on our hypergeometric functions  $F_k$ . We collect these conditions as Conjecture 5.1 and 5.2. Checking the validity of these conjectures we compute the exact form of the integral formula for the Poisson kernel on  $\mathbb{D}^4$  and  $\mathbb{D}^6$ . We also give an integral formula for the Green function in these two cases.

We will need the following representations for the classical Newtonian potentials and Poisson kernels on  $\mathbb{R}^n$ :

$$\log|x-y|^{-1} = \log|y|^{-1} + \frac{1}{2} \sum_{k=1}^{\infty} \left( \frac{|x|}{|y|} \right)^k C_k^{(0)}(\cos \theta), \quad |x| < |y|, \quad (23)$$

$$|x-y|^{2-n} = |y|^{2-n} \sum_{k=0}^{\infty} \left( \frac{|x|}{|y|} \right)^k C_k^{(\rho)}(\cos \theta), \quad |x| < |y|, n \geq 3, \quad (24)$$

$$\frac{r^2 - |x|^2}{|x-y|^n} = r^{2-n} \sum_{k=0}^{\infty} \frac{k+\rho}{\rho} \cdot \left( \frac{|x|}{r} \right)^k C_k^{(\rho)}(\cos \theta), \quad |x| < r, |y| = r. \quad (25)$$

It is easy to obtain the first two formulas using the generating function for Chebyshev and Gegenbauer polynomials respectively. The last one is the consequence of (24) and the relation  $\frac{k+\rho}{\rho} C_k^{(\rho)}(x) = C_k^{(\rho+1)}(x) - C_{k-2}^{(\rho+1)}(x)$ , where  $C_k^{(\rho)}(x)$  denotes zero if  $k$  is negative. When  $n = 2$  so  $\rho = 0$  we obtain  $k C_k^{(0)}(x) = C_k^{(1)}(x) - C_{k-2}^{(1)}(x)$ , with the same convention as before for the meaning of negative indices  $k$ . From these relations and (23) we obtain

$$4 \log|x-y|^{-1} = 4 \log|y|^{-1} - \frac{|x|^2}{|y|^2} - 2 \sum_{k=1}^{\infty} \left( \frac{|x|}{|y|} \right)^k \left( \frac{|x|^2}{|y|^2} \frac{1}{k+2} - \frac{1}{k} \right) C_k^{(1)}(\cos \theta). \quad (26)$$

To simplify our notation we introduce the following notation:

$$f_z(k) = \frac{1}{\Gamma(k + \frac{n}{2})} F_k(z).$$

It is well-known that this is an entire function of complex variable  $k$ . Observe that

$$\frac{F_k(|x|^2)}{F_k(r^2)} = \frac{f_{|x|^2}(k)}{f_{r^2}(k)}$$

for  $k = 0, 1, 2, \dots$ . We also define the function  $H_{|x|^2, r^2}(k)$  by the following equality

$$\frac{k+\rho}{\rho} \frac{f_{|x|^2}(k)}{f_{r^2}(k)} = \left( \frac{1-|x|^2}{1-r^2} \right)^\rho \left( \frac{k+\rho}{\rho} - \frac{n}{2} \frac{r^2 - |x|^2}{(1-r^2)(1-|x|^2)} \right) + \frac{r^2 - |x|^2}{(1-r^2)^{n-2}} H_{|x|^2, r^2}(k). \quad (27)$$

To proceed further we require two essential properties pertaining this function. First of all, the function  $H_{|x|^2, r^2}$  is supposed to have the following property:

$$H_{|x|^2, r^2}(k) = O(k^{-1}) \quad \text{when} \quad |k| \rightarrow \infty.$$

This property is a consequence of the following asymptotic expansion for the function  $F_k$ , which we state in the sequel as the first conjecture

**Conjecture 5.1.**

$$F(k, -\rho; k + \frac{n}{2}; z) = (1 - z)^\rho + \frac{\frac{n}{2} \rho}{k + \frac{n}{2}} z(1 - z)^{\rho-1} + O(k^{-2}). \quad (28)$$

Observe that Conjecture 5.1 holds true for even  $n$  as a consequence of the well-known relation for hypergeometric functions

$$F(\alpha, \beta; \gamma; z) = (1 - z)^{-\beta} F\left(\gamma - \alpha, \beta; \gamma; \frac{z}{z-1}\right).$$

The above relation for odd  $n$  is satisfied only for  $0 \leq z < \frac{1}{2}$  and we do not know if the above conjecture is satisfied by the function  $F_k$  for odd  $n$ .

The next conjecture, together with the first one, enable us to invert, in the sense of Laplace transform, our function  $H_{|x|^2, r^2}$ . We formulate it as follows:

**Conjecture 5.2.**  $F_k(z)$  as a function of (complex) variable  $k$  has no zeros in the region  $\{\Re(k) \geq -n/2 - \varepsilon\}$ , with  $\varepsilon = \varepsilon(z, n) > 0$ .

We now assume in the sequel that these two conjectures hold true. Then we define a function  $w_{|x|^2, r^2}(v)$  as an inverse Laplace transform (see, e.g. [F]) of the function  $H_{|x|^2, r^2}(k)$  and consequently we get

$$H_{|x|^2, r^2}(k) = \int_0^\infty e^{-kv} w_{|x|^2, r^2}(v) dv \quad (29)$$

for every complex  $k$  such that  $\Re(k) \geq -n/2 - \varepsilon$ . Now we can prove the following integral formula.

**Theorem 5.1.** Under assumptions as in Conjecture 5.1 and 5.2 we obtain

$$P_r(x, y) = \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}} r (1 - r^2)^{n-2}} \frac{r^2 - |x|^2}{|x - y|^n} \int_0^\infty \frac{w_{|x|^2, r^2}(v) L(x, y, v)}{|xe^{-v} - y|^{n-2}} dv \quad (30)$$

where

$$\begin{aligned} L(x, y, v) &= |xe^{-v/2} - ye^{v/2}|^{2\rho} \cdot \rho h(x, y, v) - |x - y|^2 [|xe^{-v/2} - ye^{v/2}|^{2\rho} - |x - y|^{2\rho}] \\ h(x, y, v) &= |xe^{-v/2} - ye^{v/2}|^2 - |x - y|^2 = |x|^2(e^{-v} - 1) + r^2(e^v - 1). \end{aligned}$$

*Proof.* Using (16) and (27) we get

$$P_r(x, y) = \mathbf{A}_1 - \mathbf{A}_2 + \mathbf{B},$$

where

$$\begin{aligned} \mathbf{A}_1 &= \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}} r^{n-1}} \left( \frac{1 - |x|^2}{1 - r^2} \right)^\rho \sum_{k=0}^\infty \frac{k + \rho}{\rho} \frac{|x|^k}{r^k} C_k^{(\rho)}(\cos \theta) \\ \mathbf{A}_2 &= \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}} r^{n-1}} \frac{n(r^2 - |x|^2)(1 - |x|^2)^{\rho-1}}{2(1 - r^2)^{\rho+1}} \sum_{k=0}^\infty \frac{|x|^k}{r^k} C_k^{(\rho)}(\cos \theta) \\ \mathbf{B} &= \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}} r^{n-1}} \frac{r^2 - |x|^2}{(1 - r^2)^{n-2}} \sum_{k=0}^\infty \frac{|x|^k}{r^k} H_{|x|^2, r^2}(k) C_k^{(\rho)}(\cos \theta). \end{aligned}$$

To deal with the series  $\mathbf{A}_1$  and  $\mathbf{A}_2$  we use (25) and (24) respectively and obtain

$$\begin{aligned}\mathbf{A}_1 &= \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}r} \left( \frac{1-|x|^2}{1-r^2} \right)^\rho \cdot \frac{r^2-|x|^2}{|x-y|^n}, \\ \mathbf{A}_2 &= \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}r} \cdot \frac{n}{2} \frac{(r^2-|x|^2)(1-|x|^2)^{\rho-1}}{(1-r^2)^{\rho+1}} \cdot \frac{1}{|x-y|^{n-2}}.\end{aligned}$$

We write  $\mathbf{B}$ , using (29), the Fubini's theorem and (24), as

$$\begin{aligned}\mathbf{B} &= \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}r^{n-1}} \cdot \frac{r^2-|x|^2}{(1-r^2)^{n-2}} \sum_{k=0}^{\infty} \frac{|x|^k}{r^k} \left( \int_0^\infty e^{-kv} w_{|x|^2, r^2}(v) dv \right) C_k^{(\rho)}(\cos \theta) \\ &= \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}r^{n-1}} \cdot \frac{r^2-|x|^2}{(1-r^2)^{n-2}} \int_0^\infty \sum_{k=0}^{\infty} \frac{|xe^{-v}|^k}{r^k} C_k^{(\rho)}(\cos \theta) w_{|x|^2, r^2}(v) dv \\ &= \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}r} \cdot \frac{r^2-|x|^2}{(1-r^2)^{n-2}} \int_0^\infty \frac{w_{|x|^2, r^2}(v) dv}{|xe^{-v} - y|^{n-2}}.\end{aligned}$$

Summing up  $\mathbf{A}_1 - \mathbf{A}_2 + \mathbf{B}$  we obtain

$$\begin{aligned}P_r(x, y) &= \frac{\Gamma(\frac{n}{2})(r^2-|x|^2)}{2\pi^{\frac{n}{2}}r(1-r^2)^{n-2}} \left[ \frac{(1-|x|^2)^\rho(1-|y|^2)^\rho}{|x-y|^n} - \frac{n(1-|x|^2)^{\rho-1}(1-|y|^2)^{\rho-1}}{2|x-y|^{n-2}} \right. \\ &\quad \left. + \int_0^\infty \frac{e^{\rho v} w_{|x|^2, r^2}(v) dv}{|xe^{-\frac{v}{2}} - ye^{\frac{v}{2}}|^{n-2}} \right].\end{aligned}$$

Recall that  $h(x, y, v) = |x|^2(e^{-v} - 1) + r^2(e^v - 1)$ . Using (29) and (27) we get

$$\begin{aligned}\int_0^\infty e^{\rho v} w_{|x|^2, r^2}(v) dv &= H(-\rho) = \frac{n}{2} (1-|x|^2)^{\rho-1} (1-r^2)^{\rho-1} \\ \int_0^\infty e^{\rho v} h(x, y, v) w_{|x|^2, r^2}(v) dv &= |x|^2 H(-\rho+1) + r^2 H(-\rho-1) + (|x|^2 + r^2) H(-\rho) \\ &= \frac{(1-|x|^2)^\rho (1-r^2)^\rho}{\rho},\end{aligned}$$

where  $H = H_{|x|^2, r^2}$  is the function defined in (27). In the last equality we use the following fact

$$|x|^2 \frac{f_{|x|^2}(-\rho+1)}{f_{r^2}(-\rho+1)} = r^2 \frac{f_{|x|^2}(-\rho-1)}{f_{r^2}(-\rho-1)}$$

which is a consequence of

$$zf_z(-\rho+1) = z \sum_{i=0}^{\infty} \frac{(-\rho+1)_i (-\rho)_i}{\Gamma(i+1)\Gamma(i+2)} z^i = \frac{1}{\rho(\rho+1)} \sum_{i=0}^{\infty} \frac{(-\rho)_{i+1} (-\rho-1)_{i+1}}{\Gamma(i+1)\Gamma(i+2)} z^{i+1} = \frac{f_z(-\rho-1)}{\rho(\rho+1)}.$$

Thus finally we get

$$\begin{aligned}P_r(x, y) &= \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}} \frac{r^2-|x|^2}{r(1-r^2)^{n-2}} \left[ \int_0^\infty \frac{\rho e^{\rho v} h(x, y, v) w_{|x|^2, r^2}(v) dv}{|x-y|^n} - \int_0^\infty \frac{e^{\rho v} w_{|x|^2, r^2}(v) dv}{|x-y|^{n-2}} \right. \\ &\quad \left. + \int_0^\infty \frac{e^{\rho v} w_{|x|^2, r^2}(v) dv}{|xe^{-\frac{v}{2}} - ye^{\frac{v}{2}}|^{n-2}} \right] \\ &= \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}r(1-r^2)^{n-2}} \frac{r^2-|x|^2}{|x-y|^n} \int_0^\infty \frac{e^{\rho v} L(x, y, v) w_{|x|^2, r^2}(v) dv}{|xe^{-\frac{v}{2}} - ye^{\frac{v}{2}}|^{n-2}} \\ &= \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}r(1-r^2)^{n-2}} \frac{r^2-|x|^2}{|x-y|^n} \int_0^\infty \frac{L(x, y, v) w_{|x|^2, r^2}(v) dv}{|xe^{-v} - y|^{n-2}}.\end{aligned}$$

This completes the proof.  $\square$

**Corollary 5.2.** If  $n = 4$  then  $w_{|x|^2, r^2}(v) = \frac{2(1+r^2)}{1-r^2} e^{-\frac{2v}{1-r^2}}$  and

$$P_r(x, y) = \frac{1}{2\pi^2 r(1-r^2)^2} \cdot \frac{r^2 - |x|^2}{|x-y|^4} \int_0^\infty \frac{[|x|^2(e^{-v} - 1) + r^2(e^v - 1)]^2 w_{|x|^2, r^2}(v)}{|xe^{-v} - y|^2} dv.$$

If  $n = 6$  then  $w_{|x|^2, r^2}(v) = w_{|x|^2, r^2}^1(v) + w_{|x|^2, r^2}^2(v)$ , where

$$\begin{aligned} w_{|x|^2, r^2}^1(v) &= 3 [1 - r^2|x|^2 + 3(r^2 - |x|^2)] e^{-bv} \cosh(cv) \\ w_{|x|^2, r^2}^2(v) &= -3 [1 + |x|^2 r^2 + 5(|x|^2 + r^2)] \frac{e^{-bv} \sinh(cv)}{2c} \end{aligned}$$

and

$$L(x, y, v) = [|x|^2(e^{-v} - 1) + r^2(e^v - 1)]^2 (2|xe^{-\frac{v}{2}} - ye^{\frac{v}{2}}|^2 + |x - y|^2),$$

hence

$$P_r(x, y) = \frac{1}{\pi^3 r(1-r^2)^4} \cdot \frac{r^2 - |x|^2}{|x-y|^6} \int_0^\infty \frac{L(x, y, v) w_{|x|^2, r^2}(v) dv}{|xe^{-v} - y|^4},$$

where  $b = \frac{7-r^2}{2(1-r^2)}$ ,  $c = c(r) = \frac{\sqrt{r^4-14r^2+1}}{2(1-r^2)}$ .

*Proof.* For  $n = 4$  we have  $F_k(z) = 1 - \frac{k}{k+2} z = (1-z) + \frac{2z}{k+2}$  (see Preliminaries) and it is obvious that Conjecture 5.1 is valid in this case. We also have

$$(k+1) \frac{F_k(|x|^2)}{F_k(r^2)} = (k+1) \frac{2+k(1-|x|^2)}{2+k(1-r^2)}$$

and it is analytic function of variable  $k$  in the region  $\{\Re(k) \geq -2 - \varepsilon\}$  for some  $\varepsilon > 0$  because the denominator has only one zero  $k_0 = -\frac{2}{1-r^2}$ . Thus we can use Theorem 5.1. We have

$$L(x, y, v) = [|x|^2(e^{-v} - 1) + r^2(e^v - 1)]^2$$

and all we have to do is to find the function  $w_{|x|^2, r^2}(v)$ . We compute it using the residue method. We have

$$\begin{aligned} \text{Res}_{k_0} H &= \lim_{k \rightarrow k_0} (k - k_0)(k+1) \frac{F_k(|x|^2)}{F_k(r^2)} \\ &= \lim_{k \rightarrow k_0} (k - k_0)(k+1) \frac{2+k(1-|x|^2)}{2+k(1-r^2)} \\ &= \frac{r^2 - |x|^2}{1 - r^2} \cdot 2 \frac{1+r^2}{1-r^2} \end{aligned}$$

and consequently

$$w_{|x|^2, r^2}(v) = 2 \frac{1+r^2}{1-r^2} e^{-\frac{2v}{1-r^2}}$$

For  $n = 6$  we have  $F_k(z) = 1 - \frac{2k}{k+3} z + \frac{k(k+1)}{(k+3)(k+4)} z^2$  and consequently

$$\frac{k+2}{2} \frac{F_k(|x|^2)}{F_k(r^2)} = \frac{k+2}{2} \cdot \frac{k(k+1)(1-|x|^2)^2 + 6k(1-|x|^2) + 12}{k(k+1)(1-r^2)^2 + 6k(1-r^2) + 12}.$$

Zeros of the dominator are given by  $k_1 = -b - c$  and  $k_2 = -b + c$ . It is easy to check that Conjecture 5.2 is valid also in this case. We have

$$L(x, y, v) = [|x|^2(e^{-v} - 1) + r^2(e^v - 1)]^2 (2|xe^{-\frac{v}{2}} - ye^{\frac{v}{2}}|^2 + |x - y|^2).$$

Observe that

$$\begin{aligned}
\text{Res}_{k_1} H + \text{Res}_{k_2} H &= \frac{(1-r^2)^4}{r^2 - |x|^2} \left[ \lim_{k \rightarrow k_1} (k - k_1) \frac{k+2}{2} \frac{F_k(|x|^2)}{F_k(r^2)} + \lim_{k \rightarrow k_2} (k - k_2) \frac{k+2}{2} \frac{F_k(|x|^2)}{F_k(r^2)} \right] \\
&= 3(1 - |x|^2 r^2 + 3(r^2 - |x|^2)), \\
\text{Res}_{k_1} H - \text{Res}_{k_2} H &= \frac{(1-r^2)^4}{r^2 - |x|^2} \left[ \lim_{k \rightarrow k_1} (k - k_1) \frac{k+2}{2} \frac{F_k(|x|^2)}{F_k(r^2)} - \lim_{k \rightarrow k_2} (k - k_2) \frac{k+2}{2} \frac{F_k(|x|^2)}{F_k(r^2)} \right] \\
&= -\frac{3}{c} [1 + |x|^2 r^2 + 5(|x|^2 + r^2)].
\end{aligned}$$

Thus we get

$$\begin{aligned}
w_{|x|^2, r^2}(v) &= \frac{1}{2} (\text{Res}_{k_1} H + \text{Res}_{k_2} H) e^{-bv} \cosh(cv) + \frac{1}{2} (\text{Res}_{k_1} H - \text{Res}_{k_2} H) e^{-bv} \sinh(cv) \\
&= w_{|x|^2, r^2}^1(v) + w_{|x|^2, r^2}^2(v).
\end{aligned}$$

The proof is completed.

Note that the polynomial  $r^4 - 14r^2 + 1$ , which appears in the definition of the function  $c(r)$ , has zero in the interval  $(0, 1)$ . More precisely we have

$$\begin{aligned}
r^4 - 14r^2 + 1 &> 0, & 0 &< r^2 < 7 - 4\sqrt{3}, \\
r^4 - 14r^2 + 1 &= 0, & r^2 &= 7 - 4\sqrt{3}, \\
r^4 - 14r^2 + 1 &< 0, & 7 - 4\sqrt{3} &< r^2 < 1.
\end{aligned}$$

If  $r^2 = 7 - 4\sqrt{3}$  we have  $c(r) = 0$  and consequently

$$w_{|x|^2, r^2}(v) = [1 - r^2|x|^2 + 3(r^2 - |x|^2)] e^{-bv} - [1 + |x|^2 r^2 + 5(|x|^2 + r^2)] \frac{v e^{-bv}}{2}.$$

In the case  $7 - 4\sqrt{3} < r^2 < 1$  the function  $c(r)$  has pure imaginary values, i.e.  $c(r) = i\tilde{c}(r)$  where

$$\tilde{c}(r) = \frac{\sqrt{|r^4 - 14r^2 + 1|}}{2(1 - r^2)}.$$

Thus we have

$$w_{|x|^2, r^2}(v) = [1 - r^2|x|^2 + 3(r^2 - |x|^2)] e^{-bv} \cos(\tilde{c}v) - [1 + |x|^2 r^2 + 5(|x|^2 + r^2)] \frac{e^{-bv} \sin(\tilde{c}v)}{2\tilde{c}}.$$

□

In the following corollary we provide the integral formula for the Green function in  $\mathbb{D}^4$ .

**Corollary 5.3.** *For  $n=4$  we get*

$$\begin{aligned}
G_D(x, y) &= \frac{1}{4\pi^2} \left( (1 - |x|^2)(1 - |y|^2) \left[ \frac{1}{|x - y|^2} - \frac{r^2}{|y|^2|x - y^*|^2} \right] \right. \\
&\quad \left. - 4 \left[ \log \frac{1}{|x - y|} - \log \frac{r}{|y||x - y^*|} \right] + (r^2 - |x|^2)(r^2 - |y|^2) \int_0^\infty \frac{w_{|x|^2, r^2}(v) dv}{|y|^2|xe^{-v} - y^*|^2} \right), \tag{31}
\end{aligned}$$

where  $w_{|x|^2, r^2}(v) = 2\frac{1+r^2}{1-r^2} e^{-\frac{2v}{1-r^2}}$  and  $y^* = \frac{r^2 y}{|y|^2}$ .

*Proof.* For  $n = 4$  we have  $G_0(z) = \frac{1}{z} + z + 2 \log z$  and  $G_k(z) = 1 - \frac{k+2}{2}z$ . Thus, using (22) for  $n = 4$ , we obtain

$$\frac{G_D(0, y)}{C_4} = \frac{G_0(|y|^2)}{|y|^2} - \frac{G_0(r^2)}{r^2} = \frac{1}{|y|^2} - |y|^2 + 4 \log |y| - \frac{1}{r^2} + r^2 - 4 \log r.$$

where  $C_4 = \frac{1}{4\pi^2}$ . Elementary algebraic computation shows that

$$\begin{aligned} \frac{G_D(0, y)}{C_4} &= \left( \frac{(1 - |x|^2)(1 - |y|^2)}{|y|^2} + \frac{|x|^2}{|y|^2} + 4 \log |y| \right) \\ &\quad - \left( \frac{(1 - |x|^2)(1 - |y|^2)}{r^2} + \frac{|x|^2|y|^2}{r^4} + 4 \log r - \frac{1 + r^2}{r^4}(r^2 - |x|^2)(r^2 - |y|^2) \right) \\ &= \mathbf{c} - \mathbf{d}. \end{aligned} \quad (32)$$

For  $n = 4$  the formula (22) takes now the form

$$\begin{aligned} \frac{G_D(x, y)}{C_4} &= \frac{G_D(0, y)}{C_4} + \frac{1}{|y|^2} \sum_{k=1}^{\infty} \left( \frac{|x|}{|y|} \right)^k (1 - \frac{k}{k+2}|x|^2)(1 - \frac{k+2}{k}|y|^2) C_k^{(1)}(\cos \theta) \\ &\quad - \frac{1}{r^2} \sum_{k=1}^{\infty} \left( \frac{|x||y|}{r^2} \right)^k (1 - \frac{k}{k+2}|x|^2)(1 - \frac{k+2}{k}r^2) \frac{(1 - \frac{k}{k+2}|y|^2)}{(1 - \frac{k}{k+2}r^2)} C_k^{(1)}(\cos \theta) \\ &= \frac{G_D(x, y)}{C_4} + \mathbf{C} - \mathbf{D} = (\mathbf{c} + \mathbf{C}) - (\mathbf{d} + \mathbf{D}). \end{aligned}$$

Put

$$\begin{aligned} f_{|y|^2, r^2}(k) &= \frac{1 - \frac{k}{k+2}|y|^2}{1 - \frac{k}{k+2}r^2}, \\ g_{|x|^2, |y|^2}(k) &= (1 - \frac{k}{k+2}|x|^2)(1 - \frac{k+2}{k}|y|^2). \end{aligned}$$

Then we have the following relations

$$\begin{aligned} f_{|y|^2, r^2}(k) &= \frac{1 - |y|^2}{1 - r^2} - 2 \frac{r^2 - |y|^2}{(1 - r^2)^2} \cdot \frac{1}{k + \frac{2}{1-r^2}}; \\ g_{|x|^2, |y|^2}(k) &= (1 - |x|^2)(1 - |y|^2) + 2 \frac{|x|^2}{k+2} - 2 \frac{|y|^2}{k}; \\ g_{|x|^2, r^2}(k) f_{|y|^2, r^2}(k) &= (1 - \frac{k}{k+2}|x|^2)(1 - \frac{k+2}{k}r^2) \cdot \frac{1 - \frac{k}{k+2}|y|^2}{1 - \frac{k}{k+2}r^2}. \end{aligned}$$

Observe also that the right-hand side of the last equality is equal to

$$\begin{aligned} &(1 - |x|^2)(1 - |y|^2) + \frac{2|x|^2 f_{|y|^2, r^2}(-2)}{(k+2)} - \frac{2r^2 f_{|y|^2, r^2}(0)}{k} - 2 \frac{(r^2 - |y|^2)}{(1 - r^2)^2} \frac{g_{|x|^2, r^2}(\frac{-2}{1-r^2})}{k + \frac{2}{1-r^2}} \\ &= (1 - |x|^2)(1 - |y|^2) + 2 \frac{|x|^2|y|^2}{r^2(k+2)} - 2 \frac{r^2}{k} - \frac{2(1+r^2)}{r^2(1-r^2)} \frac{(r^2 - |x|^2)(r^2 - |y|^2)}{k + \frac{2}{1-r^2}}. \end{aligned}$$

Thus we get  $(\mathbf{c} + \mathbf{C}) = \mathbf{C}_1 + \mathbf{C}_2$  and  $(\mathbf{d} + \mathbf{D}) = \mathbf{D}_1 + \mathbf{D}_2 - \mathbf{D}_3$  where

$$\begin{aligned}\mathbf{C}_1 &= \frac{(1 - |x|^2)(1 - |y|^2)}{|y|^2} \sum_{k=0}^{\infty} \left( \frac{|x|}{|y|} \right)^k C_k^{(1)}(\cos \theta) = \frac{(1 - |x|^2)(1 - |y|^2)}{|x - y|^2}, \\ \mathbf{C}_2 &= 4 \log |y| + \frac{|x|^2}{|y|^2} + 2 \sum_{k=1}^{\infty} \left( \frac{|x|}{|y|} \right)^k \left( \frac{|x|^2}{|y|^2} \frac{1}{k+2} - \frac{1}{k} \right) C_k^{(1)}(\cos \theta) = -4 \log \frac{1}{|x - y|}.\end{aligned}$$

The first equality follows from (24) and the other is just (26). Similarly

$$\begin{aligned}\mathbf{D}_1 &= \frac{(1 - |x|^2)(1 - |y|^2)}{r^2} \sum_{k=0}^{\infty} \left( \frac{|x||y|}{r^2} \right)^k C_k^{(1)}(\cos \theta) = \frac{(1 - |x|^2)(1 - |y|^2)r^2}{|y|^2|x - y^*|^2}, \\ \mathbf{D}_2 &= 4 \log r + \frac{|x|^2|y|^2}{r^4} + 2 \sum_{k=1}^{\infty} \frac{|x|^k|y|^k}{r^{2k}} \left( \frac{|x|^2|y|^2}{r^4} \frac{1}{k+2} - \frac{1}{k} \right) C_k^{(1)}(\cos \theta) = -4 \log \frac{r}{|y||x - y^*|}.\end{aligned}$$

To deal with  $\mathbf{D}_3$  recall that  $\int_0^\infty e^{-\frac{2v}{1-r^2}} e^{-kv} dv = \frac{1}{k + \frac{2}{1-r^2}}$ . Consequently

$$\begin{aligned}\mathbf{D}_3 &= \frac{2(1+r^2)}{r^4(1-r^2)} (r^2 - |x|^2)(r^2 - |y|^2) \sum_{k=0}^{\infty} \left( \frac{|x||y|}{r^2} \right)^k \frac{1}{k + \frac{2}{1-r^2}} C_k^{(1)}(\cos \theta) \\ &= \frac{2(1+r^2)}{r^4(1-r^2)} (r^2 - |x|^2)(r^2 - |y|^2) \sum_{k=0}^{\infty} \left( \frac{|x||y|}{r^2} \right)^k \left( \int_0^\infty e^{-\frac{2v}{1-r^2}} e^{-kv} dv \right) C_k^{(1)}(\cos \theta) \\ &= \frac{2(1+r^2)}{r^4(1-r^2)} (r^2 - |x|^2)(r^2 - |y|^2) \int_0^\infty \sum_{k=0}^{\infty} \left( \frac{|x|e^{-v}|y|}{r^2} \right)^k C_k^{(1)}(\cos \theta) e^{-\frac{2v}{1-r^2}} dv \\ &= 2 \frac{1+r^2}{1-r^2} (r^2 - |x|^2)(r^2 - |y|^2) \int_0^\infty \frac{e^{-\frac{2v}{1-r^2}} dv}{|y|^2|x e^{-v} - y^*|^2}.\end{aligned}$$

Now summing up all the components we finally obtain (31).  $\square$

It is also possible to obtain an integral formula for the Green function for  $n = 6$ .

**Corollary 5.4.** *For  $n = 6$  we have*

$$\begin{aligned}G_D(x, y) &= \frac{1}{4\pi^3} \left( (1 - |x|^2)^2(1 - |y|^2)^2 \left[ \frac{1}{|x - y|^4} - \frac{r^4}{|y|^4|x - y^*|^4} \right] \right. \\ &\quad - 6(1 - |x|^2)(1 - |y|^2) \left[ \frac{1}{|x - y|^2} - \frac{r^2}{|y|^2|x - y^*|^2} \right] \\ &\quad + 24 \left[ \log \frac{1}{|x - y|} - \log \frac{r}{|y||x - y^*|} \right] - 12 \frac{(r^2 - |x|^2)(r^2 - |y|^2)}{|y|^2|x - y^*|^2} \\ &\quad \left. + 6(r^2 - |x|^2)(r^2 - |y|^2) \int_0^\infty \frac{W_{|x|^2,|y|^2}(v) dv}{|y|^4|x e^{-v} - y^*|^4} \right),\end{aligned}$$

where

$$W_{|x|^2,|y|^2}(v) = f_1(x, y) e^{-bv} \cosh(cv) + f_2(x, y) \frac{e^{-bv} \sinh(cv)}{2(1 - r^2)c}$$

and

$$\begin{aligned} f_1(x, y) &= r^2 \frac{1+r^2}{1-r^2} (1-|x|^2)(1-|y|^2) + 2(1-|x|^2|y|^2) - 2(1-r^4) \\ f_2(x, y) &= r^2 \frac{(1+r^2)^2}{1-r^2} (1-|x|^2)(1-|y|^2) + 2(1-5r^2-2r^4)(1-|x|^2|y|^2) - 2(1-r^2)^3. \end{aligned}$$

*Proof.* We will show only some main steps of the proof of the above formula and explaining all details are left to the reader. Recall that for  $n = 6$  we have

$$F_k(z) = 1 - \frac{2k}{k+3} z + \frac{k(k+1)}{(k+3)(k+4)} z$$

and

$$G_k(z) = 1 - \frac{2(k+4)}{k+1} z + \frac{(k+3)(k+4)}{k(k+1)} z.$$

Thus we get

$$\begin{aligned} F_k(|x|^2)G_k(|y|^2) &= (1-|x|^2)^2(1-|y|^2)^2 - 6(1+|x|^2)(1+|y|^2) \left( \frac{|y|^2}{k+1} - \frac{|x|^2}{k+3} \right) \\ &\quad + 12 \left( \frac{|y|^4}{k} - \frac{|x|^4}{k+4} \right) \\ &= (1-|x|^2)^2(1-|y|^2)^2 - 6(1-|x|^2)(1-|y|^2) \left( \frac{|y|^2}{k+1} - \frac{|x|^2}{k+3} \right) \\ &\quad + 12 \left( \frac{|y|^4}{k(k+1)} - \frac{2|x|^2|y|^2}{(k+1)(k+3)} + \frac{|x|^4}{(k+3)(k+4)} \right) \end{aligned}$$

and

$$\begin{aligned} F_k(|x|^2)G_k(r^2) \frac{F_k(|y|^2)}{F_k(r^2)} &= (1-|x|^2)^2(1-|y|^2)^2 \\ &\quad - 6(1-|x|^2)(1-|y|^2) \left[ \frac{r^2}{k+1} - \frac{|x|^2|y|^2}{r^2} \frac{1}{k+3} \right] \\ &\quad + 12 \left( \frac{r^4}{k(k+1)} - \frac{2|x|^2|y|^2}{(k+1)(k+3)} + \frac{|x|^4|y|^4}{r^4} \frac{1}{(k+3)(k+4)} \right) \\ &\quad + 12(r^2-|x|^2)(r^2-|y|^2) \left[ \frac{1}{k+1} - \frac{|x|^2|y|^2}{r^4(k+3)} \right] \\ &\quad + \frac{6}{r^4}(r^2-|x|^2)(r^2-|y|^2)J(k), \end{aligned}$$

where

$$J(k) = \left[ f_1(x, y) \frac{1}{2} \left( \frac{1}{k+b+c} + \frac{1}{k+b-c} \right) + \frac{f_2(x, y)}{2c(1-r^2)} \frac{1}{2} \left( \frac{1}{k+b+c} - \frac{1}{k+b-c} \right) \right].$$

Observe also that

$$\begin{aligned} \frac{1}{|x-y|^2} &= \frac{1}{|y|^2} \sum_{k=1}^{\infty} \left( \frac{|x|}{|y|} \right)^k \left( \frac{1}{k+1} - \frac{|x|^2}{|y|^2} \frac{1}{k+3} \right) C_k^{(2)}(\cos \theta) \\ 4 \log |x-y|^{-1} &= 4 \log |y|^{-1} - \frac{2}{3} \frac{|x|^2}{|y|^2} - \frac{|x|^4}{6|y|^4} \\ &\quad + 2 \sum_{k=1}^{\infty} \frac{|x|^k}{|y|^k} \left[ \frac{1}{k(k+1)} - \frac{|x|^2}{|y|^2} \frac{2}{(k+1)(k+3)} + \frac{|x|^4}{|y|^4} \frac{1}{(k+3)(k+4)} \right] C_k^{(2)}(\cos \theta) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{r^8} \sum_{k=0}^{\infty} \frac{|x|^k |y|^k}{r^{2k}} \frac{1}{2} \left( \frac{1}{k+b+c} + \frac{1}{k+b-c} \right) C_k^{(2)}(\cos \theta) &= \int_0^{\infty} \frac{e^{-bv} \cosh (cv) dv}{|y|^4 |xe^{-v} - y^*|^4} \\ \frac{1}{r^8} \sum_{k=0}^{\infty} \frac{|x|^k |y|^k}{r^{2k}} \frac{1}{2} \left( \frac{1}{k+b+c} - \frac{1}{k+b-c} \right) C_k^{(2)}(\cos \theta) &= \int_0^{\infty} \frac{e^{-bv} \sinh (cv) dv}{|y|^4 |xe^{-v} - y^*|^4}. \end{aligned}$$

Therefore using (22) for  $n = 6$  we can obtain the desired formula.  $\square$

## References

- [BCF] P. Baldi, E. Casadio Tarabusi, A. Figá-Talamanca, *Stable laws arising from hitting distributions of processes on homogeneous trees and the hyperbolic half-plane*, Pacific J. Math. 197(2)(2001), 257-273.
- [BG] R. M. Blumenthal, R. K. Getoor, *Markov processes and potential theory*, Academic Press. Inc. (1968).
- [BJ] P. Bougerol, T. Jeulin, *Brownian bridge on hyperbolic spaces and on homogeneous trees*, Probab. Theory Related Fields 115 (1999), 95–120.
- [BGS] T. Byczkowski, P. Graczyk, A. Stos, *Poisson kernels of half-space in real hyperbolic spaces*, preprint.
- [C] I. Chavel, *Eigenvalues in Riemannian Geometry*, Academic Press. Inc. (1984).
- [Ch] K. L. Chung, *Lectures from Markov processes to Brownian Motion*, Springer-Verlag (1982).
- [ChZ] K. L. Chung, Z. Zhao, *From Brownian motion to Schrödinger's equation*, Springer-Verlag (1995).
- [D] E. B. Davies, *Heat kernels and spectral theory*, Cambridge Univ. Press (1989).
- [E] Erdelyi et al., eds., *Higher Transcendental Functions*, vol. I and II, McGraw-Hill, New York, 1953-1955.
- [Du] D. Dufresne, *The distribution of a perpetuity, with application to risk theory and pension funding*, Scand. Actuarial J., (1990), 39-79.
- [F] G. B. Folland, *Fourier Analysis and its applications*, Wadsworth and Brooks, Pacific Grove (California), 1992.
- [IW] N. Ikeda, S. Watanabe, *Stochastic differential equations and diffusion processes*, North-Holland Pub. Company (1981).
- [L] N.N Lebedev *Special functions and their applications* Dover Publications, Inc. (1972).
- [Y3] *Exponential functionals and principal values related to Brownian motion*. A collection of research papers. Edited by Marc Yor. Biblioteca de la Revista Matemática Iberoamericana, Madrid, 1997.
- [W] J. G. Wendel, *Hitting spheres with Brownian motion*, Ann. Probab. 8 164-169 (1980).